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A finite element solution technique for the Boltzmann equation

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A new method is presented for solution of the Boltzmann equation governing the dynamic behaviour of gases. The essence of the method is idealization of the problem domain into subdomains called finite elements. Then, the Galerkin assumed-mode technique is employed as the basis for discretization of the individual finite elements and also for the assembly of the resulting algebraic models for these finite elements to form an algebraic model for the complete problem. The procedure is cast in a systematic matrix notation that makes evident the broad application potential of the analysis method. An illustrative application is presented for the problem of one-dimensional, linearized Couette flow. Numerical predictions of macroscopic flow velocity and viscous shear stress based upon the subject finite element method are compared with alternative analytical and numerical results. Special attributes of the finite element method are discussed in the context of this example problem. Applications to practical problems governed by generalized forms of the Boltzmann equation are projected on the basis of concepts established herein.

1. Introduction

The kinetic theory approach to gas dynamics by way of the Boltzmann equation has obvious attraction. Solution of the Boltzmann equation for the velocity distribution function completely characterizes the flow. That is, quantities such as density, velocity, temperature and shear stress are derivable from the solution for the velocity distribution function. Moreover, the Boltzmann equation is valid over all flow régimes from free molecular to continuum.

These attractive completeness and applicability characteristics of solutions to the Boltzmann equation are of real value only in so far as such solutions are tractable. The level of difficulty, taken together with available analysis tools, has frequently prohibited direct solution of the Boltzmann equation for arbitrary Knudsen numbers.

In many gas dynamics flows of great practical interest, the complete spectrum of flow régimes from continuum to free molecular is encountered, as, for example, in the expansion of a rocket exhaust into a vacuum, or the supersonic flow of a gas past a sharp leading edge. Flows of this nature clearly indicate the need for

and usefulness of a solution technique applicable over a wide range of Knudsen numbers.

A new method, adapted from the finite element analysis technology of solid mechanics (Zienkiewicz 1967; Przemieniecki 1968), is presented herein for direct solution of the Boltzmann equation governing flow throughout all régimes from free molecular to continuum. The study described herein discloses a promising potential of the finite element method in the prediction of behaviour governed by the Boltzmann equation.

The finite element method involves a physical idealization and a mathematical discretization. In the finite element idealization of a problem domain, the continuous domain is replaced by a system of discrete component subdomains. These subdomains, hereinafter called finite elements, are defined such that appropriate inter-element contact is maintained (Tong & Pian 1967).

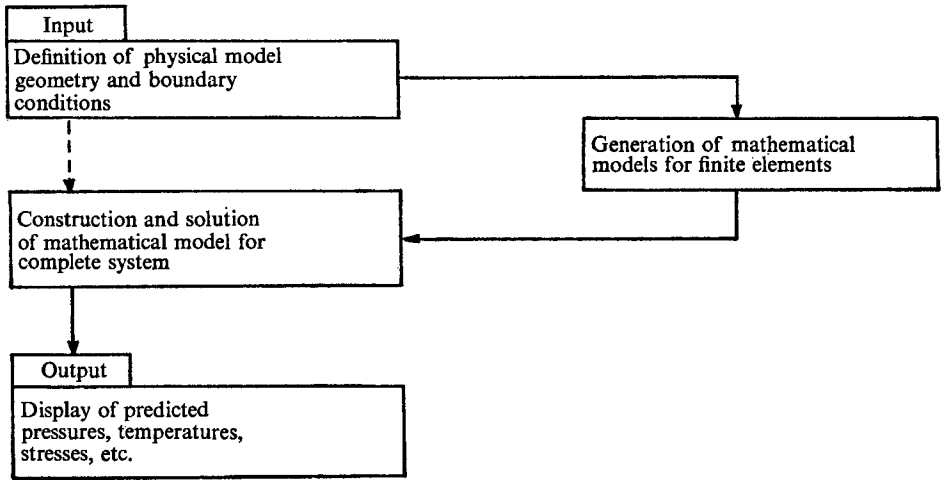

FIGURE 1. Computational flow chart for finite element formulation.

Mathematical discretization of the functions of a problem (e.g. velocity distribution function) is effected by the construction of approximating functions. Within the framework of the finite element idealization, discretization of each finite element is effected individually. Only approximating functions which provide for appropriate function continuity across interfaces with adjoining finite elements are admissible to the discretization of a finite element.

It is useful to emphasize the facility with which problematical variations are accommodated within the context of the finite element computational process. The basic computational flow of a finite element gas flow analysis is illustrated in figure 1. The important distinguishing feature to be noted in this flow chart is that the mathematical description of the system is generated independently of the objective mathematical model for the system. That is, the physical description is referred to the individual finite elements and is transformed to appropriate finite element mathematical representation without regard to the configuration of the total system and its boundary conditions. This separation contributes to

the generality of the finite element method in regard to both complexity and broad applicability.

Regarding complexity, referencing of problem description to individual finite elements permits convenient consideration of variations in geometry and boundary conditions (Mallett, Braun & Hunter 1968). Regarding applicability, this is limited only by the suitability of the finite elements made available for idealization (Oden 1969).

The particular problem considered herein is that of linearized one-dimensional Couette flow. This problem was chosen because many analytical and numerical solutions exist with which the present results may be compared. In the interest of simplicity, the Bhatnagar–Gross–Krook (1954) model of the Boltzmann equation is taken as the governing equation, and, furthermore, diffuse reflexion is assumed. As will become apparent, however, the proposed method of solution is applicable when the full collision integral is retained and when more general boundary conditions are employed.

The analytical development of the objective finite element method is outlined within §2. Numerical results derived from the finite element method are presented in §3 and compared with the predictions of alternative methods. The paper is concluded in §4 with a retrospective examination of the investigation reported herein and projection of future developments.

2. Analytical development

(i) *Basic equation*

The point of departure for this development of an analytical finite element model for the problem defined in figure 2 is taken to be the Boltzmann equation with the Bhatnagar–Gross–Krook collision term. Under the assumptions of steady flow in the absence of external forces, the governing equation for the velocity distribution function f may be written

$$\mathbf{v}.\frac{\partial f}{\partial \boldsymbol{\xi}} = \delta n(F-f), \tag{1}$$

where $\mathbf{v}$ is the microscopic velocity vector, and $\boldsymbol{\xi} = (\xi_x, \xi_y, \xi_z)$ is the position vector. The quantity δn is proportional to the local collision frequency. F is a local Maxwellian distribution defined by

$$F = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left\{-\frac{m}{2kT}(\mathbf{v}-\mathbf{u})^2\right\}, \tag{2}$$

where the macroscopic number density, velocity and temperature are given by

$$n = \iiint_{-\infty}^{+\infty} f(\boldsymbol{\xi}, \mathbf{v})\, d\mathbf{v}, \tag{3}$$

$$\mathbf{u} = \frac{1}{n}\iiint_{-\infty}^{+\infty} \mathbf{v} f(\boldsymbol{\xi}, \mathbf{v})\, d\mathbf{v}, \tag{4}$$

and

$$\frac{3kT}{2m} = \frac{1}{n}\iiint_{-\infty}^{+\infty} \tfrac{1}{2}(\mathbf{v}-\mathbf{u})^2 f(\boldsymbol{\xi}, \mathbf{v})\, d\mathbf{v}. \tag{5}$$

As shown by Gross, Jackson & Ziering (1957), the assumptions of small wall velocities $\pm \frac{1}{2}w$, constant number density n_0, and constant temperature T_0, lead to the following non-dimensional governing equation

$$c_x \frac{\partial \psi}{\partial x} + \alpha \psi - \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-c_x^2)\, \psi\, dc_x = 0. \tag{6}$$

The perturbation function ψ is related to the distribution function f by

$$f = n_0 \beta_0^3 \pi^{-\frac{3}{2}} e^{-c^2} [1 + w c_z \psi], \tag{7}$$

where $1/\beta_0 = \sqrt{(2RT_0)}$, and where R denotes the gas constant. All velocities have been non-dimensionalized by the most probably molecular speed $1/\beta_0$, and the spatial co-ordinates have been non-dimensionalized by the plate separation d. The parameter α has the physical meaning of an inverse Knudsen number based upon plate separation d.

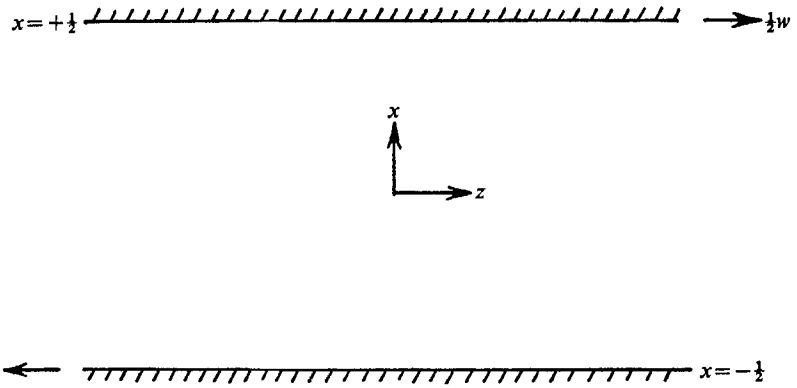

FIGURE 2. Physical problem and co-ordinate system.

Boundary conditions for (6) are obtained by assuming that molecules emitted from a surface have a Maxwellian distribution characteristic of the surface temperature and motion. It is further assumed that there is no net mass flow at the surface, and that the number of molecules between the walls is constant. With these assumptions the boundary conditions become

$$\left.\begin{aligned} \psi(x = -\tfrac{1}{2}, \quad c_x > 0) &= -1, \\ \psi(x = \tfrac{1}{2}, \quad c_x < 0) &= +1. \end{aligned}\right\} \tag{8}$$

At this point it is convenient to introduce a two stream character to the function ψ. As pointed out by Gross *et al.* (1957), such a division is necessary in order to adequately describe the distribution function within a mean free path of this surface, since in general, the function ψ will be discontinuous at solid boundaries for $c = 0$. With this assumption, the governing equation becomes

$$c_x \frac{\partial \psi^{\pm}}{\partial x} + \alpha \psi^{\pm} - \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-c_x^2)\, (\psi^{+} + \psi^{-})\, dc_x = 0, \tag{9}$$

subject to the boundary conditions

$$\psi^{\pm}(x = \mp \tfrac{1}{2}, c_x) = \mp 1, \tag{10}$$

where
$$\left.\begin{aligned}\psi^+ &= 0, \quad c_x < 0,\\ \psi^- &= 0, \quad c_x > 0.\end{aligned}\right\} \tag{11}$$

In addition to the macroscopic flow velocity u_z, defined by (5), the viscous shearing stress, defined by

$$\tau_{ij} = m\iiint_{-\infty}^{+\infty} (v_i - u_i)(v_j - u_j) f(\boldsymbol{\xi}, \mathbf{v})\, d\mathbf{v} \tag{12}$$

is of interest. In the established notation, these quantities may be written

$$\frac{u_z}{w/2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \exp(-c_x^2)(\psi^+ + \psi^-)\, dc_x, \tag{13}$$

and
$$\frac{\tau_{xz}}{\tau_{xz_{fm}}} = -\int_{-\infty}^{+\infty} \exp(-c_x^2)\, c_x(\psi^+ + \psi^-)\, dc_x, \tag{14}$$

where u_z is normalized by the non-dimensional velocity of one of the plates and τ_{xy} is non-dimensionalized by the free molecular value of the shear stress

$$(\tau_{xy_{fm}} = -w/2\sqrt{\pi}).$$

Before proceeding to the solution phase of the development it is useful to introduce a final restatement of the governing relations. Specifically, the following transformation of dependent variables is introduced in order to achieve a form characterized by homogeneous boundary conditions:

$$\phi^+(x, c) = \psi^+ - \boldsymbol{\psi}^+, \tag{15}$$

$$\phi^-(x, c) = \psi^- - \boldsymbol{\psi}^-, \tag{16}$$

where $\boldsymbol{\psi}^\pm$ denote the boundary conditions on $\psi^\pm$, respectively, and the subscript x is omitted.

The result of this set of transformations upon substitution into (9) and (10) is

$$c\frac{\partial \phi^\pm}{\partial x} + \alpha\phi^\pm - \frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-c^2}(\phi^+ + \phi^-)\, dc$$
$$= -\alpha\boldsymbol{\psi}^\pm + \frac{\alpha}{\sqrt{\pi}}\int_0^\infty e^{-c^2}\boldsymbol{\psi}^+\, dc + \frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^0 e^{-c^2}\boldsymbol{\psi}^-\, dc, \tag{17}$$

and
$$\phi^\pm(x = \mp\tfrac{1}{2}, c \gtrless 0) = 0. \tag{18}$$

This is the objective form of the governing relations. Subsequent development will focus upon the discretization of these relations within the framework of the finite element technology.

(ii) *The finite element idealization*

In the preceding section the domain of the given problem was divided into two adjoining subdomains. The finite element idealization process undertaken here is not unlike the preceding division process in that the domain is viewed as an assemblage of appropriately interconnected discrete zones called finite elements. Solution of the overall problem is approached via solution of the individual finite elements.

Figure 3 illustrates the region of the problem field that is taken to comprise a typical pair of finite elements for the adjoined positive and negative subdomains. The j^{+}th finite element in the positive subdomain d_j^{+} encompasses a physical length in x of l_j, and extends from zero to infinity over the velocity co-ordinate. The complementary jth finite element in the negative subdomain d_j^{-} encompasses a similar length in x of l_j and extends in the negative direction from zero to infinity.

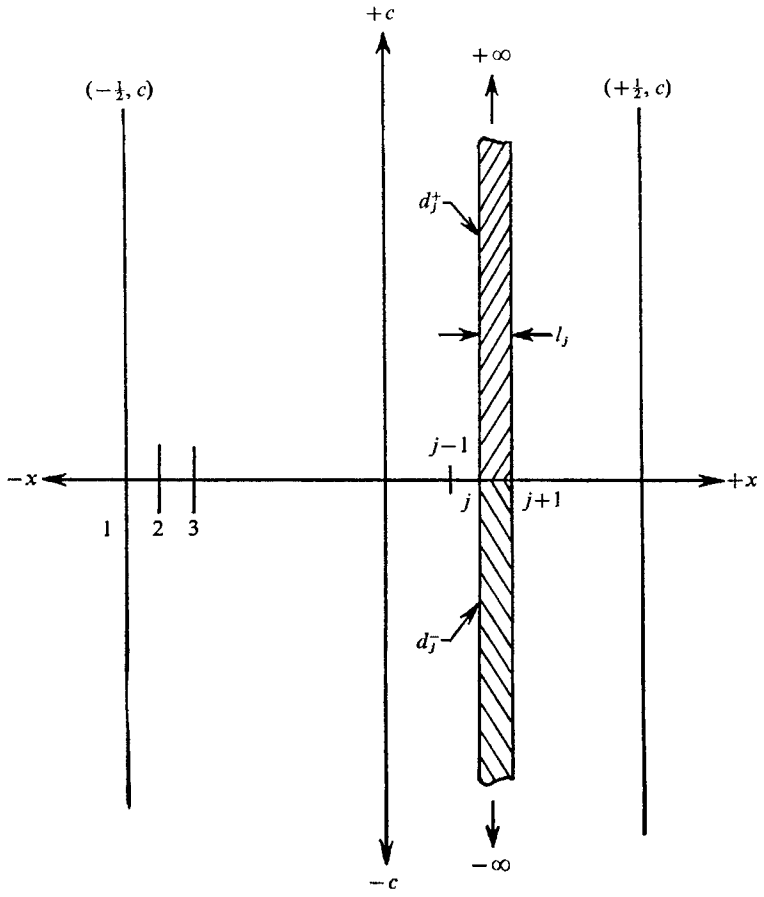

FIGURE 3. Discretization of phase space.

An approximate solution to the problem posed by (17) and (18) is sought herein using as a basis the Galerkin assumed mode technique. The point of departure in the Galerkin solution process is the construction of mode shapes with undetermined coefficients. A general form of such mode shapes for the subject problem is given by

$$\phi^{\pm}(x,c) = \sum_{r=0}^{R} \sum_{s=0}^{S} u_r(x)\, v_s(c)\, q_{rs}^{\pm}. \tag{19}$$

Admissibility requirements stipulate that these assumed functions be complete, sufficiently smooth and satisfy the boundary conditions of the problem. These requirements are generally difficult to satisfy 'in the large' but comparatively

easy to satisfy over the region of a finite element. This observation has identified the most important distinguishing characteristic of the finite element method, namely, the construction of mode shapes with respect to convenient subregions of the complete field. Thus, attention is focused upon the jth finite elements d_j^+ and d_j^-, wherein the functions ϕ_j^+ and ϕ_j^- are expressed in terms of the molecular velocity c and a local space co-ordinate η as

$$\phi_j^{\pm}(\eta, c) = (1-\eta/l) \sum_{s=0}^{S} c^s e^{-c^2} (q_{2s+1}^{\pm})_j + (\eta/l) \sum_{s=0}^{S} c^s e^{-c^2} (q_{2s+2}^{\pm})_j. \tag{20}$$

Before proceeding further, it is instructive to illustrate that the assumed functions provide for continuity between finite elements. By letting $\eta = 0$ and $\eta = l$ in the elements d_j^+ and d_j^-, one obtains

$$\phi_j^{\pm}(0, c) = \sum_{s=0}^{S} c^2 e^{-c^2} (q_{2s+1}^{\pm})_j, \tag{21}$$

and

$$\phi_j^{\pm}(l, c) = \sum_{s=0}^{S} c^s e^{-c^2} (q_{2s+2}^{\pm})_j. \tag{22}$$

This separation of the complete set for the jth element into a subset associated with $\phi_j(0, c)$ and a subset associated with $\phi_j^{\pm}(l, c)$ is illustrated in figure 4. Therein,

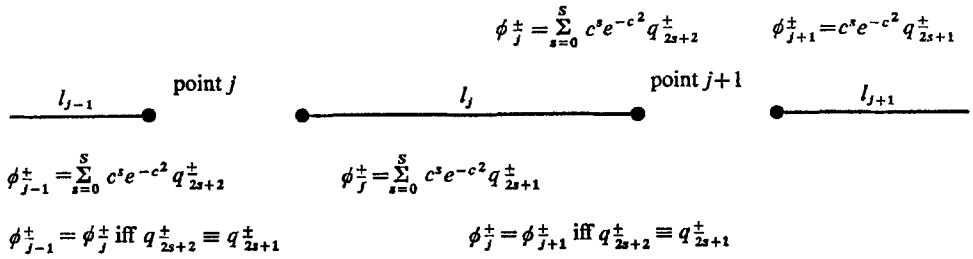

FIGURE 4. Illustration of continuity between finite elements.

the jth finite element is shown *in situ* between adjacent $j-1$ and $j+1$ elements. It follows immediately by examination of (22) and (23) together with figure 4 that the desired continuity of $\phi^{\pm}(x, c)$ is established simply by setting

$$(q_{2s+2}^{\pm})_{j-1} = (q_{2s+1}^{\pm})_j. \tag{23}$$

Moreover, it is also clear from this inspection that the homogeneous boundary conditions are satisfied simply by setting the boundary related q to zero for the finite element that resides at the boundary.

As mentioned earlier, in general the functions ϕ^+ and ϕ^- will be discontinuous at $c = 0$, and thus there are no continuity requirements to be satisfied across $c = 0$.

(iii) *The finite element representation*

Substitution of the assumed solution, (21), in the governing equation, (17), will, in general, result in an error, i.e. the approximation will not satisfy the governing equation exactly. In the finite element method, it is required that a suitable set of weighting functions be orthogonal to this error over the area of the finite element. The weighting functions chosen herein are the approximating functions

used in the expansion, which is Galerkin's method. This and alternate methods are discussed by Finlayson & Scriven (1966).

The Galerkin method for discretization of the governing integrodifferential equation proceeds as follows. Substitution of the assumed expansion ϕ_j into the integrodifferential equations yields

$$c\frac{\partial\phi_j^{\pm}}{\partial x}+\alpha\phi_j^{\pm}-\frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-c^2}(\phi_j^{+}+\phi_j^{-})\,dc+\alpha\psi^{\pm}-\frac{\alpha}{\sqrt{\pi}}\times\left[\int_0^{\infty}e^{-c^2}\psi^{+}dc+\int_{-\infty}^{0}e^{-c^2}\psi^{-}dc\right]\equiv\epsilon_j^{\pm},\quad(24)$$

where ϵ_j^{+} and ϵ_j^{-} denote the errors, or residuals, which result from the approximation given by (21). The linearly independent mode shapes are now required to be individually orthogonal to these residuals thereby leading to a set of algebraic equations governing the behaviour of the finite element. This requirement can be expressed as

$$\int_0^l\int_0^{\infty}\frac{\partial\phi^{+}}{\partial q_i}\epsilon^{+}\,dc\,d\eta=0\quad(i=1,2,\ldots,2S+2),\quad(25)$$

and

$$\int_0^l\int_{-\infty}^{0}\frac{\partial\phi^{-}}{\partial q_i}\epsilon^{-}\,dc\,d\eta=0\quad(i=1,2,\ldots,2S+2).\quad(26)$$

Note that the subscript j has been omitted.

The first step taken toward imposing these requirements is substitution for the weighting functions $\partial\phi^{\pm}/\partial q_i$ from (19). In so doing, it is convenient to separate the requirements according to whether i is odd or even to obtain the following:

$$\int_0^l\int_0^{+\infty}\left[\left(1-\frac{\eta}{l}\right)c^{\frac{1}{2}(i-1)}e^{-c^2}\right]\epsilon^{+}\,dc\,d\eta=0\quad(i\text{ odd to }2S+1),\quad(27)$$

$$\int_0^l\int_0^{+\infty}\left[\frac{\eta}{l}c^{\frac{1}{2}(i-2)}e^{-c^2}\right]\epsilon^{+}\,dc\,d\eta=0\quad(i\text{ even to }2S+2),\quad(28)$$

and

$$\int_0^l\int_{-\infty}^{0}\left[\left(1-\frac{\eta}{l}\right)c^{\frac{1}{2}(i-1)}e^{-c^2}\right]\epsilon^{-}\,dc\,d\eta=0\quad(i\text{ odd to }2S+1),\quad(29)$$

$$\int_0^l\int_{-\infty}^{0}\left[\frac{\eta}{l}c^{\frac{1}{2}(i-2)}e^{-c^2}\right]\epsilon^{-}\,dc\,d\eta=0\quad(i\text{ even to }2S+2).\quad(30)$$

The next step in proceeding toward expansion of these requirements in terms of the assumed functions is the construction of expanded expressions for the residuals. By substituting the assumed functions into the differential equations one obtains,

$$\epsilon^{\pm}=\sum_{s=0}^{S}\Bigg[-\frac{1}{l}c^{s+1}e^{-c^2}q_{2s+1}^{\pm}+\frac{1}{l}c^{s+1}e^{-c^2}q_{2s+2}^{\pm}+\alpha\left(1-\frac{\eta}{l}\right)c^{s}e^{-c^2}q_{2s+1}^{\pm}+\alpha\left(\frac{\eta}{l}\right)c^{s}e^{-c^2}q_{2s+2}^{\pm}-\frac{\alpha}{\sqrt{\pi}}\left(1-\frac{\eta}{l}\right)I_s^{+}q_{2s+1}^{+}-\frac{\alpha}{\sqrt{\pi}}\left(\frac{\eta}{l}\right)I_s^{+}q_{2s+2}^{+}-\frac{\alpha}{\sqrt{\pi}}\left(1-\frac{\eta}{l}\right)I_s^{-}q_{2s+1}^{-}-\frac{\alpha}{\sqrt{\pi}}\left(\frac{\eta}{l}\right)I_s^{-}q_{2s+2}^{-}\Bigg]-\Phi^{\pm},\quad(31)$$

where
$$\Phi^{\pm} = -\alpha\psi^{\pm} + \frac{\alpha}{\sqrt{\pi}}\int_0^{\infty} e^{-c^2}\psi^{+}\,dc + \frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-c^2}\psi^{-}\,dc, \tag{32}$$

$$I_s^{+} = \int_0^{\infty} e^{-2c^2}c^s\,dc, \tag{33}$$

and
$$I_s^{-} = \int_{-\infty}^{0} e^{-2c^2}c^s\,dc. \tag{34}$$

Development of the objective matrix equation governing behaviour within a finite element is completed simply by substitution of the expressions for the residuals into (27)–(30) and carrying out the indicated integration. This lengthy algebraic process is omitted herein and the resulting matrix equation is recorded as

$$[K]\{q\} = \{p\}, \tag{35}$$

or, in partitioned form,

$$\begin{bmatrix} K_{00}^{++} & K_{00}^{+-} & K_{0e}^{++} & K_{0e}^{+-} \\ K_{00}^{-+} & K_{00}^{--} & K_{0e}^{-+} & K_{0e}^{--} \\ K_{e0}^{++} & K_{e0}^{+-} & K_{ee}^{++} & K_{ee}^{+-} \\ K_{e0}^{-+} & K_{e0}^{--} & K_{ee}^{-+} & K_{ee}^{--} \end{bmatrix} \begin{Bmatrix} \{q_0^{+}\} \\ \{q_0^{-}\} \\ \{q_e^{+}\} \\ \{q_e^{-}\} \end{Bmatrix} = \begin{Bmatrix} \{P_0^{+}\} \\ \{P_0^{-}\} \\ \{P_e^{+}\} \\ \{P_e^{-}\} \end{Bmatrix}, \tag{36}$$

where
$$\{q_0^{-}\}^T = [q_1^{-}, q_3^{-}, \ldots, q_{S+1}^{-}], \tag{37}$$

$$\{q_0^{+}\}^T = [q_1^{+}, q_3^{+}, \ldots, q_{S+1}^{+}], \tag{38}$$

$$\{q_e^{-}\}^T = [q_2^{-}, q_4^{-}, q_6^{-}, \ldots, q_{S+2}^{-}], \tag{39}$$

and
$$\{q_e^{+}\}^T = [q_2^{+}, q_4^{+}, q_6^{+}, \ldots, q_{S+2}^{+}], \tag{40}$$

and the $[K]$ and $\{P\}$ matrices are defined in terms of the submatrices of the partitioned form of the finite element matrix representation of (36), i.e.

$$[K_{00}^{++}]_{i,j} = [-(\tfrac{1}{2})I_{i+j-1}^{+} + \alpha(\tfrac{1}{3}l)I_{i+j-2}^{+} - (\alpha/\sqrt{\pi})(\tfrac{1}{3}l)I_{j-1}^{+}J_{i-1}^{+}], \tag{41}$$

$$[K_{0e}^{++}]_{i,j} = [+(\tfrac{1}{2})I_{i+j-1}^{+} + \alpha(\tfrac{1}{6}l)I_{i+j-2}^{+} - (\alpha/\sqrt{\pi})(\tfrac{1}{6}l)I_{j-1}^{+}J_{i-1}^{+}], \tag{42}$$

$$[K_{e0}^{++}]_{i,j} = [-(\tfrac{1}{2})I_{i+j-1}^{+} + \alpha(\tfrac{1}{6}l)I_{i+j-2}^{+} - (\alpha/\sqrt{\pi})(\tfrac{1}{6}l)I_{j-1}^{+}J_{i-1}^{+}], \tag{43}$$

$$[K_{ee}^{++}]_{i,j} = [+(\tfrac{1}{2})I_{i+j-1}^{+} + \alpha(\tfrac{1}{3}l)I_{i+j-2}^{+} + (\alpha/\sqrt{\pi})(\tfrac{1}{3}l)I_{j-1}^{+}J_{i-1}^{+}], \tag{44}$$

$$[K_{00}^{+-}]_{i,j} = [-(\alpha/\sqrt{\pi})(\tfrac{1}{3}l)I_{j-1}^{-}J_{i-1}^{+}], \tag{45}$$

$$[K_{0e}^{+-}]_{i,j} = [-(\alpha/\sqrt{\pi})(\tfrac{1}{6}l)I_{j-1}^{-}J_{j-1}^{+}], \tag{46}$$

$$[K_{e0}^{+-}]_{i,j} = [K_{0e}^{+-}]_{i,j}, \tag{47}$$

$$[K_{ee}^{+-}]_{i,j} = [K_{00}^{+-}]_{i,j}, \tag{48}$$

$$\{P_0^{+}\}_i = \{\tfrac{1}{2}l\Phi^{+}J_{i-1}^{+}\}, \tag{49}$$

$$\{P_e^{+}\}_i = \{P_0^{+}\}_i, \tag{50}$$

where
$$J_j^{+} = \int_0^{\infty} e^{-c^2}c^i\,dc, \tag{51}$$

and
$$J_i^{-} = \int_{-\infty}^{0} e^{-c^2}c^i\,dc. \tag{52}$$

The remaining submatrices in (36) are obtained simply by interchanging the '$-$' and '$+$' superscripts.

Having given explicit definition to the matrices of (35), development of the objective finite element representation is completed by the construction of explicit relations for the flow velocity and shear stress. These follow immediately upon substitution of the assumed mode shapes as

$$\begin{aligned}\begin{pmatrix} u_z \\ \tfrac{1}{2}w \end{pmatrix} = {} & \frac{1}{\sqrt{\pi}}(J_0^- \Psi^- + J_0^+ \Psi^+) \\ & + \frac{1}{\sqrt{\pi}}\left(1 - \frac{\eta}{l}\right)([I_u^+]\{q_0^+\} + [I_u^-]\{q_0^-\}) \\ & + \frac{1}{\sqrt{\pi}}\left(\frac{\eta}{l}\right)([I_u^+]\{q_e^+\} + [I_u^-]\{q_e^-\}),\end{aligned} \tag{53}$$

$$\begin{aligned}\tau_{yz}/\tau_{xy_{fm}} = {} & -(J_1^- \Psi^- + J_1^+ \Psi^+) \\ & -\{1 - (\eta/l)\}([I_\tau^+]\{q_0^+\} + [I_\tau^-]\{q_0^-\}) \\ & -(\eta/l)([I_\tau^+]\{q_e^+\} + [I_\tau^-]\{q_e^-\}),\end{aligned} \tag{54}$$

where

$$[I_u^\pm]^T = [I_0^\pm, I_1^\pm, \ldots, I_S^\pm], \tag{55}$$

$$[I_\tau^\pm]^T = [I_1^\pm, I_2^\pm, \ldots, I_{S+1}^\pm]. \tag{56}$$

Upon solution for the primary variables $\{q\}$ these relations permit calculation of numerical values for the non-dimensional velocity and shear stress within the finite element subdomain.

(iv) *The complete system representation*

The preceding development has made available at this point a matrix relation which governs the behaviour within a finite element of arbitrary physical length. The assumed mode shapes inherent within this matrix representation of the finite element permit only a linear variation in $\psi(x, c)$ in the x-direction. A piecewise linear approximation of the actual $\psi(x, c)$ is thereby achieved by subdivision of the x-interval into multiple finite elements. Of course, the greater the number of elements used, the better this approximation will be.

The matrix representation developed for the finite element embodies an arbitrary order of function in the c-direction. Thus, refinement with respect to the c-direction can be carried out independent of the subdivision of the physical dimension into individual finite elements.

Given a particular breakdown into finite elements, say 3, and a prescribed number of terms in the c-direction, the associated set of finite element matrices can be constructed and placed in an inflated matrix relation of the form

$$\begin{bmatrix} [K]^{(1)} & & \\ & [K]^{(2)} & \\ & & [K]^{(3)} \end{bmatrix} \begin{Bmatrix} \{q\}^{(1)} \\ \{q\}^{(2)} \\ \{q\}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \{p\}^{(1)} \\ \{p\}^{(2)} \\ \{p\}^{(3)} \end{Bmatrix}. \tag{57}$$

The deficiency of the relation, as regards solution of the given analysis problem, is that connexion of the finite elements to establish the necessary piecewise

continuity of the functions $\phi^+(x, c)$ and $\phi^-(x, c)$ in the x-direction and satisfy the boundary conditions has not been effected. The desired continuity is established through the mechanism of a connectivity matrix. The connectivity matrix is a Boolean transformation between the complete set of variables $\{q\}$ of (57) and the independent variables of the assembled system $\{q_c\}$. This transformation matrix is readily derived by observation of the finite element idealization to obtain

$$\{q\} = [\Gamma_c]\{q_c\}, \tag{58}$$

where the nature of this relation is made clear by its explicit definition for the subject problem in figure 5.

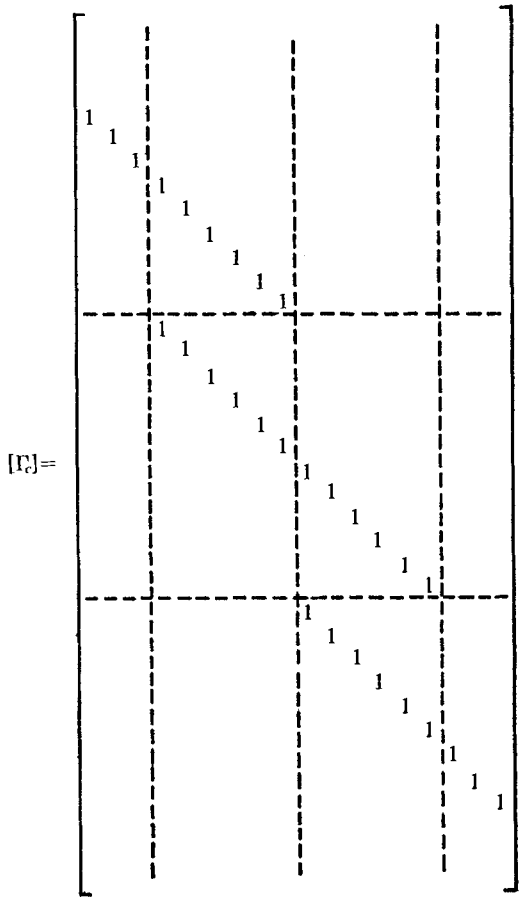

FIGURE 5. The connectivity matrix Γ_c.

The connectivity defined by (58) is imposed by application to (57) to obtain

$$[K_c]\{q_c\} = \{P_c\}, \tag{59}$$

where

$$[K_c] = [\Gamma_c]^T [K][\Gamma_c] \tag{60}$$

and

$$\{P_c\} = [\Gamma_c]^T \{P\}. \tag{61}$$

Equation (59) is the objective form of the finite element based matrix relation governing behaviour of the subject problem. Solution for the $\{q_c\}$ yields a functional approximation to the functions $\phi^{\pm}(x, c)$. Analytical integrations of these approximate functions then yield the non-dimensional velocity and shear stress profiles through (53) and (54).

In the present study, the width of the finite elements l was taken to be 0·1 for values of α in the range $0{\cdot}01 \leqslant \alpha \leqslant 10{\cdot}0$. Both the cases $S = 2$ and $S = 3$ were considered. To extend the results accurately into the continuum régime ($10 \leqslant \alpha \leqslant 50$) the case $S = 2$ and $l = 0{\cdot}05$ was also considered. All computations were performed on an IBM 360–65 electronic computer using a matrix inversion technique.

3. Results and discussions

In §3 the results of the finite element method are compared with the numerically exact solution of Willis (1962), the half-range moment solution of Gross *et al.* (1957), the moment solution of Lees (1959) and the discrete ordinate solution of Huang & Giddens (1967).

For the purpose of comparison, it is convenient to consider the flow régimes $0{\cdot}01 < \alpha \leqslant 5$ and $\alpha > 5$ separately. Such a division is necessary because of the nature of the approximations inherent in the present solution technique. In the first régime, a linear approximating function adequately represents the physical space dependence of the distribution function. In this régime, the solution must be improved by increasing the order of the approximation in velocity space. This is also a characteristic of the velocity moment and discrete ordinate methods, and one would thus expect the present results to be comparable to the solutions obtained by these techniques. In the second régime, a low-order approximation in velocity space suffices, but the physical space representation of the distribution function must be improved, either by using higher order approximation functions or by refining the physical space subdivision. The latter method was used in the present study.

Tables 1 and 2 present a comparison of the modified slip velocity $(1 - 2q_w/w)$, and shear stress ratio $\tau_{xy}/\tau_{xy_{fm}}$, for $l = 0{\cdot}1$ and $S = 2$ and 3 with the results obtained by alternate methods. For $0{\cdot}01 \leqslant \alpha \leqslant 5$, the present results are in good agreement with both the numerical solution of Willis and the discrete ordinate solution of Huang & Giddens and are superior to the analytical solutions of Gross, Jackson & Ziering and of Lees. In the second régime ($\alpha > 5$), the present theory tends to underestimate both the velocity slip and shear stress. However, as may be seen in table 3, refinement of the physical space subdivision improves the present solution.

4. Conclusions

The analysis of the preceding sections has indicated that very accurate solutions to the Boltzmann equation for the problem of linearized Couette flow may be obtained by the finite element method. The solution yields results comparable

to both the numerical solution of Willis and the discrete ordinate solution of Huang & Giddens and superior to the analytical solutions of Gross, Jackson & Ziering and of Lees.

α	Willis	Gross Jackson Ziering	Lees	Huang Giddens	Finite element $S = 2$, $l = 0{\cdot}1$	Finite element $S = 3$, $l = 0{\cdot}1$
0·01	0·9731	0·9871		0·9787	0·9819	0·9811
0·1	0·8559	0·8844		0·8526	0·8574	0·8501
1·0	0·4962	0·4871	0·6393	0·4963	0·4942	0·4963
1·25	0·4515	0·4422	0·5864	0·4515	0·4507	0·4503
1·50	0·4151	0·4064	0·5416	0·4151	0·4149	0·4135
1·75	0·3846		0·5032	0·3846	0·3847	0·3834
2·0	0·3586	0·3523	0·4698	0·3587	0·3588	0·3561
2·5	0·3165		0·4149	0·3166	0·3166	0·3138
3·0	0·2837	0·2808	0·3714	0·2838	0·2835	0·2811
3·5	0·2572		0·3362	0·2574	0·2568	
4·0	0·2354	0·2341	0·3070	0·2355	0·2347	0·2329
5·0	0·2015	0·2008	0·2617	0·2016	0·2003	0·1992
7·0	0·1566	0·1564	0·2020	0·1567	0·1545	0·1546
10·0	0·1174	0·1174	0·1506	0·1176	0·1149	0·1151

TABLE 1. Comparison of modified slip velocity for $l_k = 0{\cdot}1$, $S = 2$ and 3 with solutions obtained using alternate methods

α	Willis	Gross Jackson Ziering	Lees	Huang Giddens	Finite element $S = 2$, $l = 0{\cdot}1$	Finite element $S = 3$, $l = 0{\cdot}1$
0·01	0·9913	0·9913	0·9944	0·9913	0·9913	0·9912
0·1	0·9253	0·9215	0·9466	0·9254	0·9245	0·9243
1·0	0·6008	0·5899	0·6393	0·6007	0·6060	0·5999
1·25	0·5517	0·5427	0·5864	0·5511	0·5555	0·5494
1·50	0·5099	0·5025	0·5416	0·5096	0·5138	0·5079
1·75	0·4745	0·4787	0·5032	0·4742	0·4781	0·4735
2·0	0·4440	0·4391	0·4698	0·4436	0·4493	0·4414
2·5	0·3938	0·3904	0·4149	0·3933	0·3962	0·3912
3·0	0·3539	0·3516	0·3714	0·3535	0·3557	0·3515
4·0	0·2946	0·2934	0·3070	0·2943	0·2953	0·2924
5·0	0·2425	0·2517	0·2617	0·2522	0·2525	0·2506
7·0	0·1964	0·1960	0·2020	0·1963	0·1954	0·1949
10·0	0·1474	0·1472	0·1506	0·1473	0·1456	0·1457

TABLE 2. Comparison of shear stress ratio for $l_k = 0{\cdot}1$, $S = 2$ and 3 with solutions obtained using alternate methods

In comparing the proposed technique with existing methods, factors such as computational effort, accuracy, and computational speed should be considered.

Regarding computational effort, the present technique is similar to the discrete ordinate method in that a single formulation yields a solution for an arbitrary order of approximation. This is in contrast to the velocity moment methods, where higher-order solutions are increasingly difficult to obtain.

Concerning accuracy, the finite element solution appears to converge quite rapidly in the near free molecule régime. Examination of tables 1 and 2 shows that the second- and third-order solutions agree well with the exact solution of Willis. Very accurate results may also be obtained in the near continuum régime, as is shown in table 3.

	Shear stress		Slip velocity	
α	Huang & Giddens	Finite element $s = 2, l = 0{\cdot}05$	Huang & Giddens	Finite element $s = 2, l = 0{\cdot}05$
10	0·1473	0·1475	0·8824	0·8830
20	0·08044	0·07981	0·9348	0·9369
30	0·05533	0·05470	0·9558	0·9569
40	0·042317	0·04155	0·9664	0·9673
50	0·03406	0·03349	0·9728	0·9737

TABLE 3. Comparison of slip velocity and shear stress with discrete ordinate solution for large values of α

A discussion of computational time is difficult, since this topic is either not applicable to the above theories, i.e. they are analytical, or is not mentioned by the various authors. A typical run time for the present technique was approximately 5 min (real time) on an IBM 360–65, with most of the time being used for the matrix inversion. It should be noted, however, that the numerical technique utilized out-of-core storage, which increased the real time considerably. Furthermore, no attempt was made to take advantage of the symmetry of the problem, which would have reduced the size of the matrix to be inverted, and thus the computational time. Preliminary work on a numerical technique which considers the hyperbolic nature of the governing equations indicates that a substantial reduction in computational time may be possible.

Although the present study is based upon a particularly simple model of the Boltzmann equation, the technique can be applied to more complex governing equations. One such equation is the linearized Boltzmann equation in a curvilinear co-ordinate system, where derivatives of the distribution function with respect to microscopic velocities are introduced. Preliminary results for the problem of linearized cylindrical Couette flow are encouraging.

The extension of the technique to non-linear problems is straightforward. An appropriate functional expansion, e.g. of the type proposed by Mintzer (1965), coupled with an iterative technique, will yield an algebraic system of equations no more complex than the one considered herein. An analysis of the problem of non-linear Couette flow with heat transfer based on the BGK model requires a single matrix inversion (for a given Knudsen number), with the iterations requiring only matrix multiplications. For this problem, the computational time for present technique should be comparable to that required by the discrete ordinate method.

To extend the present technique to unsteady problems, it is only necessary to allow the coefficients q to be functions of time. For unsteady Couette flow,

an analysis identical to the one presented herein yields a system of equations of the form,

$$[M_c]\left\{\frac{dq_c}{dt}\right\} + [K_c]\{q_c\} = \{P_c\},$$

which must be solved by an appropriate technique for initial value problems.

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